Leçon 08 – Correction des "Exercez-vous"

Exercez vous 4

Calculer les dérivées partielles secondes des fonctions de l'exercice précédent :

f : $(x, y) \in \mathbb{R}^2 \mapsto x^2 - xy$ g : $(x, y) \in \mathbb{R}^* \times \mathbb{R}^* \mapsto (x^2+1) / xy$ h : $(x, y) \in \mathbb{R}_+^* \times \mathbb{R} \mapsto x^y$

(on pourra écrire $x^y = e^{y \ln x}$)

Solution

f : pour tout $(x, y) \in \mathbb{R}^2$ $\dfrac{\partial f}{\partial x}(x,y) = 2x - y$ $\dfrac{\partial f}{\partial y}(x,y) = -x$, d'où

$\dfrac{\partial^2 f}{\partial x^2}(x,y) = \dfrac{\partial}{\partial x}(2x - y) = 2$

$\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial}{\partial y}(2x - y) = -1$, et bien sûr $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y)$

$\dfrac{\partial^2 f}{\partial y^2}(x,y) = \dfrac{\partial}{\partial y}(-x) = 0.$

g: pour tout (x,y) de $\mathbb{R}^* \times \mathbb{R}^*$, $\dfrac{\partial g}{\partial x}(x,y) = \dfrac{x^2 - 1}{x^2 y} = \dfrac{1}{y} - \dfrac{1}{x^2 y}$ $\dfrac{\partial g}{\partial y}(x,y) = -\dfrac{x^2 + 1}{xy^2}$

d'où $\mathbf{\dfrac{\partial^2 g}{\partial x^2}} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{y} - \dfrac{1}{x^2 y}\right) = \mathbf{\dfrac{2}{x^3 y}}$ $\mathbf{\dfrac{\partial^2 g}{\partial y^2}} = \dfrac{\partial}{\partial y}\left(-\dfrac{x^2 + 1}{xy^2}\right) = \mathbf{\dfrac{2x^2 + 2}{xy^3}}$

$\mathbf{\dfrac{\partial^2 g}{\partial y \partial x}(x,y) = \dfrac{\partial^2 g}{\partial x \partial y}(x,y)} = \dfrac{\partial}{\partial y}\left(\dfrac{x^2 - 1}{x^2 y}\right) = \mathbf{\dfrac{-x^2 + 1}{x^2 y^2}}$.

h : pour tout (x,y) de $\mathbb{R}_+^* \times \mathbb{R}$, $\dfrac{\partial h}{\partial x}(x,y) = y\, x^{y-1}$, $\dfrac{\partial h}{\partial y}(x,y) = \ln(x)\, x^y$,

d'où $\mathbf{\dfrac{\partial^2 h}{\partial x^2}(x,y)} = \dfrac{\partial}{\partial x}(y\, x^{y-1}) = \mathbf{y(y-1)\, x^{y-2}}$

$\mathbf{\dfrac{\partial^2 h}{\partial y^2}(x,y)} = \dfrac{\partial}{\partial y}(\ln(x)\, e^{y \ln(x)}) = (\ln(x))^2 e^{y \ln(x)} = \mathbf{(\ln(x))^2\, x^y}$

$\mathbf{\dfrac{\partial^2 h}{\partial y \partial x}}(x,y) = \mathbf{\dfrac{\partial^2 h}{\partial x \partial y}(x,y)} = \dfrac{\partial}{\partial x}(\ln(x)\, x^y) = x^{y-1} + \ln(x)\, y\, x^{y-1} = \mathbf{(1 + \ln(x))\, x^{y-1}}$