Intégrale

$X = x+2$

1. $$\int_{-2}^{-1} \frac{(x+1)(x^2+3x+3)}{(x^2+4x+5)^2}\,dx = \int_0^1 \underbrace{\frac{(X-1)(X^2-X+1)}{(X^2+1)^2}}_{:= f(X)}\,dX$$

On fait un DES

$$f(X) = \frac{aX+b}{1+X^2} + \frac{cX+d}{(1+X^2)^2}$$

$$a = 1 = \lim_{X\to\infty} X f(X)$$

$$\text{I} \quad b+d = -1 \quad (X=0)$$

$$\text{II} \quad \frac{a+b}{2} + \frac{c+d}{4} = 0 \quad (X=1)$$

$$\text{III} \quad \frac{-a+b}{2} + \frac{-c+d}{4} = -\frac{3}{2}$$

$\text{II}+\text{III}$: $b + \frac{d}{2} = -\frac{3}{2}$ $\quad b = -2$

$b + d = -1$ $\quad d = 1$

$c = 1$

$$f(X) = \frac{X-2}{1+X^2} + \frac{X+1}{(1+X^2)^2}$$

$$\int_0^1 \frac{X-2}{1+X^2}\,dX = \left[\frac{1}{2}\ln(1+X^2) - 2\,\mathrm{Arctan}(X)\right]_0^1$$

$$\int_0^1 \frac{X+1}{(1+X^2)^2} = \left[-\frac{1}{2}\frac{1}{(1+X^2)}\right]_0^1 + \int_0^1 \frac{dX}{(1+X^2)^2} \longrightarrow \text{cf poly.}$$

2. $$\int_0^{\pi/4} \frac{\sin(x)}{1+\cos(x)+\cos(2x)}\,dx = \int_0^{\pi/4} \frac{\sin(x)\,dx}{1+\cos(x)+2\cos^2(x)-1}$$

$y = \cos(x) \qquad dy = -\sin(x)\,dx$

$$= \int_{\sqrt{2}/2}^{1} \frac{dy}{(2y+1)\,y}$$

$$= \int_{\sqrt{2}/2}^{1} \left(\frac{1}{y} - \frac{2}{2y+1}\right) dy$$

$$= \Big[\ln(y) - \ln(2y+1)\Big]_{\sqrt{2}/2}^{1}.$$

3. $$\int_0^{\pi/2} \frac{\cos(x)\,dx}{\sin^2(x) - 5\sin(x) + 6} = \int_0^1 \frac{dy}{y^2-5y+6} = \int_0^1 \frac{dy}{(y-2)(y-3)}$$

$$= \int_0^1 \frac{1}{y-3} - \frac{1}{y-2} = \left[\ln\left(\frac{|y-3|}{|y-2|}\right)\right]_0^1$$

$$= \ln(2) - \ln\frac{3}{2}.$$

4. $$\int_0^1 \frac{dx}{\operatorname{ch} x} = \int_0^1 \frac{2e^x\,dx}{e^{2x}+1} = 2\int_1^e \frac{dy}{y^2+1} = 2\Big[\mathrm{Arctan}\,y\Big]_1^e$$

5 $$\int_0^1 \frac{e^x+1}{e^{2x}+1}\,dx = \int_0^1 \frac{y+1}{y(y^2+1)}\,dy = \int_1^e \frac{a}{y} + \frac{by+c}{1+y^2}\,dy \quad (a, b, c \text{ à déterminer})$$

$$= a + \frac{b}{2}\Big[\ln(1+y^2)\Big]_1^e + c\,\Big[\mathrm{Arctan}\,y\Big]_1^e.$$

Exercice 2 (Primitive)

1. $f(x) = \dfrac{x^4+x^2+2}{(x+1)(x+2)^2} = x-5+\dfrac{18(x+1)}{(x+2)^2}$

R $f(x) = x-5+18\left(\dfrac{1}{x+2} - \dfrac{1}{(x+2)^2}\right)$

Primitive $F(x) = \dfrac{x^2}{2} - 5x + 18\ln|x+2| + \dfrac{1}{x+2}$

2. $\cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

$$\int \frac{dx}{\cos x} = \int \frac{1+\tan^2(x/2)}{1-\tan^2(x/2)}\,dx \qquad X = \tan(x/2)$$

$$= \int \frac{2\,dX}{1-X^2} = \int \frac{1}{1-X} + \frac{1}{1+X}\,dX$$

$$= \ln(|X-1|\,|X+1|)$$

Primitive de $\dfrac{1}{\cos}$: $x \longmapsto \ln\left(\left|1-\tan^2\left(\frac{x}{2}\right)\right|\right)$

3. $$\int \frac{\tan x}{1+\cos x}\,dx = \int \frac{\sin x\,dx}{(\cos x)(1+\cos x)}$$

$$= \int \frac{-dy}{y(y+1)} \qquad y = \cos x .$$

$$= \int \frac{1}{y+1} - \frac{1}{y}$$

Primitive : $x \longmapsto \ln\left|\dfrac{1+\cos x}{\cos x}\right|$

4. $$\int \frac{e^{-2x}dx}{1+e^{-x}} = -\int \frac{y\,dy}{1+y} \qquad y = e^{-x} .$$

$$= -\int 1 - \frac{1}{1+y}\,dy$$

$$= \int \frac{1}{1+y} - 1$$

Primitive $x \mapsto \ln(1+e^{-x}) - e^{-x}$.

Exercice 3

$f(x) = \dfrac{\sin x}{1+\cos^2(x)}$

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx .$$

$$= \int_0^1 \frac{dy}{1+y^2} + \int_{-1}^0 \frac{dy}{1+y^2} = 2\int_0^1 \frac{dy}{1+y^2}$$

$$= 2\,\mathrm{Arctan}\,1 = \frac{\pi}{2} .$$

Ex 4

1: $$I_{n+1} = \int_0^1 (1-t^2)(1-t^2)^n\,dt .$$

$$= I_n - \int_0^1 t^2(1-t^2)^n\,dt$$

$$= I_n + \left\{ \left[t\,\frac{(1-t^2)^{n+1}}{2\ n+1} \right]_0^1 - \int_0^1 \frac{(1-t^2)^{n+1}}{2(n+1)} \right\}$$

$$= I_n - \frac{I_{n+1}}{2(n+1)}$$

$$I_{n+1}\left(1+\frac{1}{2(n+1)}\right) = I_n \Rightarrow I_{n+1} = \frac{2n+2}{2n+3} I_n$$

$$I_0 = 1$$

$$I_n = \prod_{k=0}^{n-1} \frac{2k+2}{2k+3}$$

3. $$I_n = \int_0^1 \sum_{k=0}^{n} C_n^k (-1)^k t^{2k}$$

$$= \sum_{k=0}^{n} C_n^k (-1)^k \int_0^1 t^{2k} dt = \sum_{k=0}^{n} C_n^k \frac{(-1)^k}{2k+1}.$$

Intégrales Généralisées

1) $0 \leq \frac{e^{-x}}{x^2} \leq \frac{1}{x^2} \quad \forall x \geq 1$ et $\int_1^{+\infty} \frac{dx}{x^2} < +\infty$ (Riemann)

2) $\frac{x + x^\alpha}{\sqrt{x} + x^3} \underset{+\infty}{\sim} \frac{x^{\max(\alpha;1)}}{x^3}$ intégrable si $\max(\alpha;1) < 2$

$\underset{0}{\sim} \frac{x^{\min(\alpha;1)}}{\sqrt{x}}$ intégrable si $\min(\alpha;1) > -\frac{1}{2}$

soit $-\frac{1}{2} < \alpha < 2$

3) $\frac{1-e^{-x}}{x^{3/2}} \underset{+\infty}{\sim} \frac{1}{x^{3/2}}$ donc intégrable sur $[0;+\infty[$.

$\underset{0}{\sim} \frac{1}{\sqrt{x}}$ (Rq $x \longmapsto \frac{1-e^{-x}}{x^{3/2}}$ continue donc intégrable sur tout intervalle $[a;b]$ $0<a<b<+\infty$).

4) $\left|\frac{\ln(1+x)\cos x}{x^4 + x^{4/3}}\right| \underset{+\infty}{\sim} \frac{|\cos(x)|\,|\ln(x)|}{x^4} = o\left(\frac{1}{x^3}\right)$ donc intégrable en $+\infty$.

$\underset{0}{\sim} \frac{1}{x^{1/3}}$ donc intégrable en 0.

5. $\int_2^{\infty} \frac{x^\alpha \, dx}{\ln x}$ diverge si $\alpha \geq 0$.

suppossons $\underline{\alpha \leq 0}$ $\quad \alpha = -\beta \quad \beta > 0$.

$$\int_2^{+\infty} \frac{dx}{x^\beta \ln(x)} \longrightarrow \text{intégrable si } \beta > 1 \text{ (Rieman)}$$

$\searrow$ si $\beta = 1$ diverge (CDV $y = \ln(x)$)

6. $\int_0^{\infty} \frac{dt}{t^{3/4}\sqrt{|1-t|}}$

$$\frac{1}{t^{3/4}\sqrt{|1-t|}} \underset{0}{\sim} \frac{1}{t^{3/4}} \text{ intégrable.}$$

$$\underset{+\infty}{\sim} \frac{1}{t^{3/4+1/2}} \text{ intégrable.}$$

⚠ il faut faire l'étude en $\underline{\underline{1}}$

$$\int_1^{\infty} \frac{dt}{t^{3/4}\sqrt{|1-t|}}$$

$$\frac{1}{t^{3/4}\sqrt{|1-t|}} \underset{t \to 1^+}{\sim} \frac{1}{\sqrt{t-1}} \text{ intégrable (Riem).}$$

Même chose si $0 < t < 1$.

Ex 2

1. $\left|\frac{\sin x}{x^{\alpha+1}}\right| \leq \frac{1}{x^{\alpha+1}}$ intégrable (Rieman et $\alpha > 0$)

$$\int_1^M \frac{\cos(x)}{x^\alpha} = \left[\frac{\sin(x)}{x^\alpha}\right]_1^M + \alpha \int_1^M \frac{\sin x}{x^{\alpha+1}} \, dx.$$

Quand $M \to \infty$, on a.

$$\int_1^{\infty} \frac{\cos x}{x^\alpha} \, dx = -\sin(1) + \alpha \int_1^{\infty} \frac{\sin x}{x^{\alpha+1}} \, dx.$$

2.

$$\int_1^{\infty} \frac{\cos^2(x)}{x} \, dx \geq \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\cos^2(x) \, dx}{x}$$

$$\geq \sum_{k=1}^{\infty} \int_0^{\pi} \frac{\cos^2(x+k\pi) \, dx}{x+k\pi}$$

$$\geq \sum_{k=1}^{\infty} \frac{1}{(k+1)\pi} \int_0^{\pi} \cos^2(x) \, dx. = +\infty.$$

Ensuite $|\cos x| \leq 1 \Rightarrow |\cos(x)| \geq \cos^2(x)$

D'où $\quad \frac{|\cos x|}{x} \geq \frac{\cos^2(x)}{x}$.

diverge.

Exercice 3

1.) $D_\Gamma = \mathbb{R}_*^+$

2.) $\int_0^\infty e^{-t} t^{x-1} dt \overset{IPP}{=} \cancel{\left[e^{-t} \frac{t^x}{x} \right]_0^\infty} + \int_0^\infty e^{-t} \frac{t^x}{x} dt$

D'où $\Gamma(x+1) = x \, \Gamma(x)$.

3.) $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.

$\Gamma(n) = n!$