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Exercice 1

1. $f_1(x) = \dfrac{x - x^{3/2}}{1+x^2-\sqrt{x}}$ $\qquad f_1(x) \underset{+\infty}{\sim} \dfrac{-x^{3/2}}{x^2} = -\dfrac{1}{\sqrt{x}}$

$f_1(x) \underset{0}{\sim} x$.

2. $f_2(x) = e^x + \cos(x)$ $\qquad f_2(x) \underset{+\infty}{\sim} e^x \quad f_2(x) \underset{0}{\sim} 2$

3. $f_3(x) = e^x - x^{3/2}$ $\qquad f_3(x) \underset{+\infty}{\sim} e^x \quad f_3(x) \underset{0}{\sim} 1$

4. $f_4(x) = \sqrt{x} + \sqrt{\sin x} + \sqrt{\ln(1+x)}$ $\qquad f_4(x) \underset{+\infty}{\sim} \sqrt{x}$

$f_4(x) \underset{0}{\sim} 3\sqrt{x}$

Exercice 2

1. $f_1(x) = \dfrac{(1+x^2)\sin(x)\,(e^{2x}-1)}{\tan(5x^2)} \underset{0}{\sim} \dfrac{x \times 2x}{5x^2} = \dfrac{2}{5}$

2. $f_2(x) = \dfrac{x \ln(1+x)}{\mathrm{Arctan}^2(x)} \underset{0}{\sim} \dfrac{x \times x}{x^2} = 1$

3. $f_3(x) = e^{\frac{1}{\tan^2(x)} \ln(\cos x)} = e^{\frac{-x^2/2 + o(x^2)}{x^2+o(x^2)}} \longrightarrow e^{-1/2}$

4. $f_4(x) = (\pi - 2x)\tan(x) = 2\dfrac{\sin(x)}{\dfrac{\cos(x)}{\frac{\pi}{2}-x}} \underset{x\to\pi/2}{\longrightarrow} \dfrac{2\sin\pi/2}{-\cos'(\pi/2)} = 2$

5. $f_5(x) = \dfrac{\ln(x)}{\cos(\frac{\pi}{2}x)} \longrightarrow \dfrac{\ln'(1)}{\cos(\frac{\pi}{2}\cdot)'(1)} = \dfrac{1}{-\pi/2} = -\dfrac{2}{\pi}$

6 $f_6(x) = \dfrac{\sqrt{1+x^2}}{\sin(1/x)}\left(e^{\frac{1}{x}}-1\right) \sim \dfrac{x \times \frac{1}{x}}{1/x} \underset{+\infty}{\sim} x \longrightarrow +\infty$.

7. $f_7(x) = e^{\ln(x)\ln\left(1+\frac{\ln x}{x}\right)} = e^{\ln(x)\left(\frac{\ln(x)}{x} + o\left(\frac{\ln x}{x}\right)\right)}$

$= e^{\frac{\ln^2(x)}{x} \underset{+\infty}{+} o\left(\frac{\ln^2(x)}{x}\right)} \longrightarrow 1$.

8- $f_8(x) = e^{\frac{x}{2}\ln\left(\frac{1-\frac{1}{x^2}}{1+1/x^2}\right)} = e^{\frac{x}{2}\left(-\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right)} \longrightarrow 1$

Exercice 3

1. $f_1(x) = x^3 - 3x^2 + 3$

$f_1(2) = -1$

$f_1'(2) = 3(2)^2 - 6(2) = 0$

$f_1''(2) = 6(2) - 6 = 6$

$f_1'''(2) = 6 \qquad f_1^{(k)}(2) = 0$ si $k > 3$

$f_1(x) = -1 + o(1) \quad n=0$

$f_1(x) = -1 + o(x) \quad n=1$

$f_1(x) = -1 + 6\dfrac{(x-2)^2}{2!} + \dfrac{6(x-2)^3}{3!} + o((x-2)^3)$

$= -1 + 3(x-2)^2 + (x-2)^3 + o((x-2)^3)$

$f_1(x) = -1 + 3(x-2) + (x-2)^3 + o((x-2)^3)$.

2.) $\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + o(x^8)$

$f_2(x) = -\dfrac{1}{3!} + \dfrac{x^2}{5!} + o(x^3)$

3) $f_3(x) = \dfrac{x}{\tan(x)} = \dfrac{x}{x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6)} = \dfrac{1}{1 + \frac{x^2}{3} + \frac{2x^4}{15} + o(x^5)}$

$= 1 - \left(\dfrac{x^2}{3} + \dfrac{2x^4}{15} + o(x^5)\right) + \left(\dfrac{x^2}{3} + \dfrac{2x^4}{15} + o(x^5)\right)^2 + o(x^5)$

$= 1 - \dfrac{x^2}{3} + \left(\dfrac{1}{9} - \dfrac{2}{15}\right)x^4 + o(x^5)$

4. $f_4(x) = e^x \ln(x-1)$

$= e^{2+h}\ln(1+h) \qquad (h = x-2)$

$= e^2\left(1+h+\frac{h^2}{2}+\frac{h^3}{6}+o(h^3)\right)\left(h-\frac{h^2}{2}+\frac{h^3}{3}+o(h^3)\right)$

$= e^2\left[h+\frac{h^2}{2}+\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{2}\right)h^3+o(h^3)\right]$

$= e^2\left[(x-2)+\frac{(x-2)^2}{2}+\left(\frac{x-2)^3}{3}\right)+o((x-2)^3)\right]$

5. $f_5(x) = \sqrt{1+1-\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^4)}$

$=\sqrt{2}\sqrt{1-\frac{x^2}{4}+\frac{x^4}{48}+o(x^4)}$

$=\sqrt{2}\left(1+\frac{1}{2}\left(-\frac{x^2}{4}+\frac{x^4}{48}+o(x^4)\right)-\frac{1}{8}\left(-\frac{x^2}{4}+o(x^3)\right)^2\right)$

$=\sqrt{2}\left(1-\frac{x^2}{8}+\left(\frac{1}{96}-\frac{1}{64}\right)x^4+o(x^4)\right)$

6. $f_6(x) = e^{x^2\ln\left(\frac{\sin 1/x}{1/x}\right)}$

$= e^{x^2\ln\left(1-\frac{1}{3!}\frac{1}{x^2}+\frac{1}{5!}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right)}$

$= e^{x^2\left[-\frac{1}{6x^2}+\frac{1}{5!x^4}-\frac{1}{2}\frac{1}{(3!)^2}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]}$

$= e^{-\frac{1}{6}}\, e^{\left(\frac{1}{5!}-\frac{1}{2(3!)^2}\right)\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)}$

$= e^{-\frac{1}{6}}\left(1+\left(\frac{1}{5!}-\frac{1}{2(3!)^2}\right)\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)$

Exercice 4

1. $y = 1+2(x-1)$

2. La courbe C_f est sous la tangente.

Exercice 5 $\qquad x = 1+h$

1. $f(x) = \sqrt{x} = \sqrt{1+h}$

$= 1+\frac{h}{2}-\frac{h^2}{8}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2.3}h^3+o(h^3)$

$= 1+\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+o(h^3)$

$g(x) = e^{\sqrt{1+h}}$

$= e^{1+\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+o(h^3)}$

$= e^1\left(1+u(h)+\frac{u(h)^2}{2}+\frac{u(h)^3}{6}+o(h^3)\right)$

avec $u(h) = \frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+o(h^3)$

puis remplacer et ne garder que les monômes de degré ≤ 3.

Exercice 6

$f_1(x) = \frac{x^3+3}{x+1} = x^2\left(1+\frac{3}{x^3}\right)\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+o(x^{-3})\right)$

$= x^2\left(1-\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}+o\left(\frac{1}{x^3}\right)\right)$

$f_2(x) = \frac{\ln(1+x)}{\cos x-1} = \frac{x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)}{-x^2/2+\frac{x^4}{4!}+o(x^4)} = -\frac{2}{x}\,\frac{\left(1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)\right)}{1-\frac{x^2}{12}+o(x^2)}$

$= -\frac{2}{3x}\left(1-\frac{x}{2}+\left(\frac{1}{3}+\frac{1}{12}\right)x^2+o(x^2)\right)$

Exercice 6

3. $f_3(x) = \ln(1+x^2) - \frac{1}{x}$

$$= \ln(x^2) + \ln\left(1+\frac{1}{x^2}\right) - \frac{1}{x}$$

$$= 2\ln(x) - \frac{1}{x} + \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$$

Exercice 7

1.) $$\lim_{x\to 0} \frac{x - \operatorname{Arcsin}(x)}{\sin^3(x)} = \lim_{x\to 0} \frac{-x^3/3!}{x^3} = -\frac{1}{6}$$

2) $$\lim_{x\to 0} \frac{e^{\sqrt{4+x^2}} - e^{\sqrt{4-x^2}}}{\tan^2(x)} = \frac{e^{2\sqrt{1+\frac{x^2}{4}}} - e^{2\sqrt{1-\frac{x^2}{4}}}}{\tan^2(x)}$$

$$= \frac{e^{2}\left(1+\frac{x^2}{4} - \left(1-\frac{x^2}{4}\right) + o(x^2)\right)}{x^2 + o(x^2)}$$

$$\xrightarrow[x\to 0]{} \frac{e^2}{2}$$

3) $$\frac{x^e - e^x}{(x-e)^2} = \frac{e^{e\ln(e+h)} - e^{e+h}}{h^2} \quad (h = x-e)$$

$$= e^{e+h}\left(\frac{e^{-\frac{h^2}{2e}} - 1}{h^2}\right) \longrightarrow e^e \times \left(-\frac{1}{2e}\right)$$

4) $$\left(\frac{\sin x}{\operatorname{sh} x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{x^2}\left(\frac{x - x^3/3! + o(x^3)}{x + x^3/3! + o(x^3)}\right)}$$

$$= e^{\frac{1}{x^2}\left(\frac{1 - x^2/6 + o(x^2)}{1 + x^2/6 + o(x^2)}\right)} \longrightarrow +\infty.$$

Exercice 8

1.) $f(x) = \sqrt{x^2+1} - x$.

$$= |x|\sqrt{1+\frac{1}{x^2}} - x.$$

$$= |x|\left(1 + \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right) - x.$$

$$= |x| - x + \frac{1}{2|x|} + o\left(\frac{1}{|x|}\right).$$

$$= -2\min(x, 0) + \frac{1}{2|x|} + o\left(\frac{1}{|x|}\right).$$

2.) $g(x) = x^3 \sin\left(\frac{1}{x}\right)$

$$= x^3\left(\frac{1}{x} - \frac{1}{3!x^3} + o\left(\frac{1}{x^4}\right)\right)$$

$$= x^2 - \frac{1}{6} + o\left(\frac{1}{x}\right).$$